With a known functional relationship between the X_{NM} and U_{LK}, one has the following linkage equations between the two-dimensional force and strain tensors:

$$x_{lK} = \int_{b_-}^{b_+} (\partial U_{NM}/\partial u_{lK}) X_{NM} B_1 B_2 dt_3,$$

$$x_{33} = \int_{b_-}^{b_+} (\partial U_{NM}/\partial u_{33}) X_{NM} B_1 B_2 dt_3, \quad (4.9)$$

$$z_{lK} = \int_{b_-}^{b_+} (\partial U_{NM}/\partial w_{lK}) X_{NM} B_1 B_2 dt_3.$$

Equations (4.1)-(4.9) form a closed system for the unknown functions u_N, v_n, u_{33}, u_{nM}, w_{nM}, x_{NM}, z_{nM} and their first-order partial derivatives. The bending tensor v_{nM} plays an auxiliary role in this system of abbreviated denotation for differential expression (4.2).

When the two-dimensional system of (4.1)-(4.9) has been solved, the three-dimensional parameters of the state of stress and strain in the shell are determined from the scheme presented in Sec. 3.

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EXISTENCE OF SOLUTIONS IN IDEAL HENCKE PLASTICITY

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The existence of a weak solution in the theory of ideal Hencke plasticity is obtained only in the particular case of the Mises flow condition and under the assumption of isotropy of the material [1]. The strain vector is here found from a space conjugate to $L^\infty(\Omega)$. The existence of a solution for an arbitrary flow condition and without the assumption of isotropy is proved in this paper. The displacement vector belongs to the space $L^{3/2}(\Omega)$.

The governing equations of the plasticity theory under consideration yield a representation of the total strains in the form of a sum of elastic and plastic components

$$\varepsilon_{ij}(u) = c_{ijkl}\sigma_{kl} + \xi_{ij}, \; i, j = 1, 2, 3, \quad (1)$$

where the stresses do not exceed the yield point $\Phi(\sigma) \leqslant 0$, while the plastic strains ξ_{ij} satisfy the inequality [1-3]

$$\xi_{ij}(\tau_{ij} - \sigma_{ij}) \leqslant 0 \;\; \forall\tau, \; \Phi(\tau) \leqslant 0. \quad (2)$$

The equilibrium equations are satisfied in the domain $\Omega \subset R^3$

$$-\sigma_{ij,j} = f_i, \; i = 1, 2, 3. \quad (3)$$

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On the boundary of the domain Γ the condition

$$u = 0 \tag{4}$$

is valid. Here f_i are given mass forces, and the function Φ describes the flow condition. It is assumed that Φ is continuous, convex, and the set $\tilde{K} = \{\lambda \in R^6 | \Phi(\lambda) \leqslant 0\}$ contains zero as an inner point. The boundary Γ is considered smooth. All the quantities with two subscripts are symmetric $\sigma = \{\sigma_{ij}\}$, $\varepsilon_{ij}(u) = (1/2)(u_{i,j} + u_{j,i})$, $u = (u_1, u_2, u_3)$, $c_{ijkl} \in L^\infty(\Omega)$, $\lambda = \{\lambda_{ij}\}$, the tensor c_{ijkl} possesses the usual symmetry and positive-definiteness properties; summation is assumed to be over repeated subscripts. Also let $K = \{\sigma \in L^2(\Omega) | \sigma(x) \in \tilde{K}$ almost everywhere in $\Omega\}$. The follow result holds.

THEOREM. Let $f_i \in L^3(\Omega)$, $i = 1, 2, 3$ and let a solution σ^0 of the system (3) exist such that for a certain constant $\delta > 0$ the inclusion $(1 + \delta)\sigma^0 \in K$ will be valid. Then there are functions σ and u satisfying (3) where

$$\sigma \in K:\ C(\sigma, \tau - \sigma) + \int_\Omega u_i (\tau_{ij,j} - \sigma_{ij,j})\, dx \geqslant 0 \quad \forall \tau \in K \cap V,$$
$$\sigma_{ij,j} \in L^3(\Omega), \quad u \in L^{3/2}(\Omega). \tag{5}$$

Here

$$C(\sigma, \tau) = \int_\Omega c_{ijkl}\sigma_{kl}\tau_{ij} dx, \quad V = \{\sigma \in L^2(\Omega) \mid \sigma_{ij,j} \in L^3(\Omega)\}.$$

The inequality (5) is the relations (1), (2), (4) written in a form corresponding to the existing regularity of the solution.

Proof. The scheme of the discussion is the following. First we examine the auxiliary problem with a penalty, and by using reciprocity methods we prove the existence of approximate solutions. We then obtain a priori estimates that are uniform in the penalty parameter $\varepsilon > 0$, and we then pass to the limit.

Let π be the projection operator in R^6 with the customary Euclidean norm in the set $\tilde{K}$. We define a functional in the space $L^2(\Omega)$ ($\varepsilon > 0$ is fixed)

$$H(\sigma) = \frac{1}{2} C(\sigma, \sigma) + P(\sigma), \quad P(\sigma) = \frac{1}{2\varepsilon} \| \sigma - \pi\sigma \|_2^2$$

and the closed convex sets

$$A = \{\sigma \in L^2(\Omega) | \sigma_{ij,j} = f_i,\ i = 1, 2, 3\},\ A_0 = \{\sigma \in L^2(\Omega) | \sigma_{ij,j} = 0,\ i = 1, 2, 3\}.$$

Using the first Korn inequality in $H_0^1(\Omega)$, it can be proved that the set

$$W = \{e_{ij} \mid e_{ij} \in L^2(\Omega),\ e_{ij} = \varepsilon_{ij}(u),\ u = (u_1, u_2, u_3) \in H_0^1(\Omega)\}$$

is closed in $L^2(\Omega)$.

Furthermore, let us set

$$F(e) = \begin{cases} -\langle f_i, u_i \rangle, & \text{if} \quad e_{ij} \in W, \qquad e = \{e_{ij}\}, \\ +\infty & \text{otherwise}, \end{cases}$$
$$G(e) = \sup_{\tau \in L^2(\Omega)} \{\langle \tau, e \rangle - H(\tau)\}, \quad \tau = \{\tau_{ij}\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega)$. Because of the closedness of W the functional F is weakly semicontinuous from below.

Let us consider the problem

$$\inf_{e \in L^2(\Omega)} \{G(e) + F(e)\} = \inf_{u \in H_0^1(\Omega)} \left\{ \sup_{\tau \in L^2(\Omega)} [\langle \tau, \varepsilon(u) \rangle - H(\tau)] - \langle f_i, u_i \rangle \right\}. \tag{6}$$

We prove that this problem has a solution. Indeed, we have at the point $\tau = \bar{\tau}$ where the exact upper bound is reached

$$\langle H'(\bar{\tau}), \sigma \rangle - \langle \varepsilon(u), \sigma \rangle = 0 \ \forall \sigma \in L^2(\Omega).$$

Consequently

$$\varepsilon_{ij}(u) = c_{ijkl}\bar{\tau}_{kl} + P'(\bar{\tau})_{ij}, \quad P'(\bar{\tau})_{ij} = (1/\varepsilon)(\bar{\tau} - \pi\bar{\tau})_{ij}. \tag{7}$$

Here and above the prime denotes the derivatives of the appropriate functionals. We substitute the value obtained $\varepsilon_{ij}(u)$ into the square bracket in (6) for $\tau = \bar{\tau}$. We obtain that the value of this bracket equals $(1/2)C(\bar{\tau}, \bar{\tau}) - P(\bar{\tau}) + \langle P'(\bar{\tau}), \bar{\tau}\rangle$. However, it follows from (7) [c is independent of u, $\bar{\tau}$, $\|\cdot\|_p$ is the norm in $L^p(\Omega)$]

$$\|u\|_{H_0^1(\Omega)} \leqslant c\left(\|\bar{\tau}\|_2 + \|P'(\bar{\tau})\|_2\right).$$

Moreover, from the convexity of the functional P we obtain that $P(\bar{\tau}) \leqslant \langle P'(\bar{\tau}), \bar{\tau}\rangle$. Therefore, the value of the braces in (6) tends to $+\infty$ for $\|u\|_{H_0^1(\Omega)} \to +\infty$ [we emphasize that ε is still fixed so that $\|P'(\bar{\tau})\|_2 \leqslant c\|\bar{\tau}\|_2$]. The coercivity of the functional (6) is set. Hence, the existence of the solution $u = u^\varepsilon \in H_0^1(\Omega)$ follows from the weak semicontinuity from below.

We now construct the problem conjugate to (6). We have $G^*(\tau) = H(\tau)$ and

$$F^*(-\tau) = \sup_{e\in L^2(\Omega)} \{\langle \tau, e\rangle - F(e)\} = \sup_{u\in H_0^1(\Omega)} \{\langle \tau, \varepsilon(u)\rangle + \langle f_i, u_i\rangle\} = \psi_A(\tau).$$

Here $\psi_A(\tau)$ is the display function of the set A, i.e., a function equal to zero in the set A and $+\infty$ outside. Therefore, the following [4]

$$\inf_{\tau\in L^2(\Omega)} \{G^*(\tau) + F^*(-\tau)\} = \inf_{\tau\in A} H(\tau) \tag{8}$$

will be a problem reciprocal to (6) relative to the perturbation $\Lambda(e, q) = G(e) + F(e - q)$. This problem has a solution which is characterized by the inequality $\sigma = \sigma^\varepsilon \in A : \langle H'(\sigma^\varepsilon), \tau - \sigma^\varepsilon\rangle \geqslant 0\ \forall\tau \in A$. We here substitute $\tau = \sigma^\varepsilon \pm \tau_0, \tau_0 \in A_0$. We obtain

$$\langle H'(\sigma^\varepsilon), \tau_0\rangle = 0 \quad \forall\tau_0 \in A_0. \tag{9}$$

The identity (9) will play a substantial part in obtaining a priori estimates of the solution.

The solutions $\sigma^\varepsilon \in L^2(\Omega)$ and $u^\varepsilon \in H_0^1(\Omega)$ of the conjugate problems (6) and (8) are related to the extremal relationships [4]

$$G(\varepsilon(u^\varepsilon)) + G^*(\sigma^\varepsilon) = \langle \varepsilon(u^\varepsilon), \sigma^\varepsilon\rangle.$$

Substituting the values of G and G*, we find

$$H(\sigma^\varepsilon) - \langle \varepsilon(u^\varepsilon), \sigma^\varepsilon\rangle \leqslant H(\tau) - \langle \varepsilon(u^\varepsilon), \tau\rangle \quad \forall\tau \in L^2(\Omega).$$

It hence follows that

$$\varepsilon_{ij}(u^\varepsilon) = c_{ijkl}\sigma^\varepsilon_{kl} + P'(\sigma^\varepsilon)_{ij}. \tag{10}$$

We now obtain the a priori estimates of the solutions on the basis of (9) and (10). We substitute as τ_0 in the identity (9) the quantity $\bar{\sigma} = \sigma^\varepsilon - \sigma^0$, σ^0 is the solution of (3). We will have

$$C(\bar{\sigma}^\varepsilon, \bar{\sigma}^\varepsilon) + \langle P'(\sigma^\varepsilon), \bar{\sigma}^\varepsilon\rangle = -C(\sigma^0, \bar{\sigma}^\varepsilon).$$

Since $\sigma^0 \in K$, then $\langle P'(\sigma^\varepsilon), \bar{\sigma}^\varepsilon\rangle \geqslant 0$, and this means that from this relationship there follows

$$\|\bar{\sigma}^\varepsilon\|_2 \leqslant c, \quad \langle P'(\sigma^\varepsilon), \bar{\sigma}^\varepsilon\rangle \leqslant c. \tag{11}$$

The constants c are here independent of ε.

Furthermore, it follows from the conditions of the theorem that $(1+\delta)\sigma^0 \in K$. Hence, there is a constant δ_0 dependent on δ and $\tilde{K}$ for which $\operatorname{dist}(\sigma^0(x), C\tilde{K}) \geqslant \delta_0$; $C\tilde{K}$ is the supplement to the set $\tilde{K}$. We assume that for $x \in \Omega$ the imbedding $\sigma^\varepsilon(x) = \sigma^0(x) + \bar{\sigma}^\varepsilon(x) \in C\tilde{K}$ is valid. Then the hyperplane is $\{\xi \in R^6 | [\xi, l(x)] = b\}$, where b is a certain number while

$$l(x) = P'(\sigma^\varepsilon)(x)|P'(\sigma^\varepsilon)(x)|^{-1} \tag{12}$$

separates the sets $\sigma^\varepsilon(x)$ and $\tilde{K}$. Here $[\cdot, \cdot]$ denotes the scalar product corresponding to the Euclidean norm. Since $\sigma^\varepsilon(x) \in C\tilde{K}$, $\sigma^0(x) + \delta_0 l(x) \in \tilde{K}$, $0 \in \tilde{K}$, then the following inequalities hold:

$$[\sigma^0(x)+\bar{\sigma}^\varepsilon(x),\ l(x)]\geqslant b,\ [\sigma^0(x)+\delta_0 l(x),\ l(x)]\leqslant b.$$

We hence obtain $[\bar{\sigma}^\varepsilon(x), l(x)]\geqslant\delta_0$. Consequently, by multiplying (12) scalarly by $\bar{\sigma}^\varepsilon$, we will have

$$|P'(\sigma^\varepsilon)(x)|\leqslant\delta_0^{-1}[P'(\sigma^\varepsilon)(x),\bar{\sigma}^\varepsilon(x)].$$

This inequality is satisfied also in the case $\sigma^\varepsilon(x)\in\tilde{K}$. Integrating it over the domain Ω and taking account of (11), we obtain

$$\|P'(\sigma^\varepsilon)\|_1\leqslant c$$

with constant c independent of ε. By using this inequality, we obtain $\|\varepsilon(u^\varepsilon)\|_1\leqslant c$ from (10). It is proved in [5] that if $\Omega\subset R^3$ is a domain with regular boundaries and $\varphi=(\varphi_1,\varphi_2,\varphi_3)\in H_0^1(\Omega)$, then there exists a constant $c>0$ independent of φ for which

$$\|\varphi\|_{3/2}\leqslant c\sum_{i,j=1}^{3}\|\varepsilon_{ij}(\varphi)\|_1.$$

Therefore, the estimate

$$\|u^\varepsilon\|_{3/2}\leqslant c \tag{13}$$

with constant independent of ε is valid.

According to (11) and (13), it can be considered that there exists a subsequence denoted in the previous manner that possesses the property

$$\sigma^\varepsilon\to\sigma \text{ weakly in } L^2(\Omega),\ u^\varepsilon\to u \text{ weakly in } L^{3/2}(\Omega). \tag{14}$$

We multiply (10) by $\tau_{ij}-\sigma_{ij}^\varepsilon$, $\tau\in K\cap V$ and we obtain

$$C(\sigma^\varepsilon,\tau-\sigma^\varepsilon)+\int_\Omega u_i^\varepsilon(\tau_{ij,j}-\sigma_{ij,j}^\varepsilon)\,dx\geqslant 0.$$

Taking into account that $\sigma_{ij,j}^\varepsilon=-f_i$ as well as the fact that $\varliminf_{\varepsilon\to 0} C(\sigma^\varepsilon,\sigma^\varepsilon)\geqslant C(\sigma,\sigma)$, we go over here to the lower limit as $\varepsilon\to 0$, on the basis of (14). Consequently, we obtain the inequality (5). The imbedding $\sigma\in\tilde{K}$ is proved by standard reasoning [1]. The theorem is proved completely.

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